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Numerical solution of two-parameter singularly perturbed boundary value problems via exponential spline

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Abstract In this paper, a singularly perturbed semi-linear boundary value problem with two-parameters is considered. The problem is solved using exponential spline on a Shishkin mesh. The convergence analysis is derived and the method is convergent independently of the perturbation parameters. Numerical results are presented which support the theoretical results.

1. Introduction

It is well known that in various fields of science and engineering many reaction–diffusion as well as convection–diffusion problems naturally occur. Heat transfer with large Péclet numbers, nuclear engineering, combustion, control theory, elasticity, fluid mechanics, aerodynamics, quantum mechanics, optimal control, chemical-reactor theory, convection–diffusion process and geophysics are some examples of these fields. However, reaction–diffusion problem when $\varepsilon_d = 0$ and convection–diffusion problem when $\varepsilon_d = 1$ are enclosed in the two-parameter singularly perturbed boundary value problem (Gracia et al., 2006; Kadalbajoo and Gupta, 2009; Kadalbajoo and Yadaw, 2008; Lin et al., 2009; Linß and Roos, 2004; Rao and Chakravarthy, 2012; Rao and Kumar, 2008; Rao et al., 2010; Reddy and Pramod, 2003; Roos and Uzelac, 2003; Stynes and Kopteva, 2011; Valanarasu and Ramanujam, 2003; Valarmathi and Ramanujam, 2003). This two-parameter singularly perturbed semi-linear boundary value problem has the following form:

$$Ly(x) \equiv -\varepsilon_d y'' + \varepsilon_c p(x)y' + f(x,y) = 0, \quad x \in (0,1), \tag{1}$$

where the boundary conditions are:

$$y(0) = \mu_1, \quad y(1) = \mu_2, \tag{2}$$

with two small parameters $0 < \varepsilon_c << 1$, $0 < \varepsilon_d << 1$, where $p(x)$ and $f(x,y)$ are sufficiently smooth functions and $p(x) \geqslant \tilde{p} > 0$, for $x \in [0,1]$, assuming that $f_y(x,y) > 0$.

Different numerical methods were proposed to solve singularly perturbed problem with $\varepsilon_c = 1$ and $\varepsilon_c = 0$ such as Reddy and Pramod (2003), Kadalbajoo and Gupta (2009), Lin et al. (2009), Rao and Kumar (2008), Rao and Chakravarthy (2012), Rao et al. (2010) and Stynes and Kopteva (2011). On the other hand, the solution of the two-parameter singular perturbation problem was made in limited researches such as Gracia et al. (2006), Valanarasu and Ramanujam (2003) and Valarmathi and Ramanujam (2003). Linß and Roos (2004) considered linear two-parameter singularly perturbed convection–diffusion problem and used the simple upwind-difference scheme on Shishkin mesh to establish almost first-order convergence, independently of the parameters ε_c and ε_d. Roos

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and Uzelac (2003) also considered linear two-parameter singularly perturbed boundary value problem. They used stream-line diffusion finite element method on properly chosen Shishkin mesh. As a result they proved almost second-order pointwise convergence uniformly with respect to the parameters ε_c andε_d. B-spline collocation method for solving linear two-parameter singularly perturbed boundary value problems on piecewise-uniform Shishkin mesh was investigated by Kadalbajoo and Yadaw (2008), they conclude the uniform convergence of the second order.

Herein, the exponential spline difference scheme method is used to solve the two-parameter singularly perturbed boundary value problems given by Eqs. (1) and (2) by utilizing a properly chosen piecewise-uniform Shishkin mesh proving that the method is uniformly convergent independently of parameters ε_c and ε_d.

This paper is organized as follows: Section 2 provides a priori estimates of the continuous problem. In Section 3, the Shishkin mesh technique is introduced. The design of exponential spline difference scheme method is presented in Section 4 followed by the uniform convergence of the method in Section 5. In Section 6, numerical results and the comparison of approximate solutions are presented to demonstrate the uniformity of the convergence. Finally, Section 7 is devoted for the conclusions.

2. Properties of the continuous problem

The construction of a layer-adapted mesh as well as the analysis of the method requires information about the behavior of derivatives of the exact solution, where we substitute $f(x,y) = r(x)y - g(x)$ into Eq. (1). To describe the layers at $x = 0$ and $x = 1$, the characteristic equation is used as given in Linß and Roos (2004), as follows:

$$-\varepsilon_d\eta(x)^2 + \varepsilon_c p(x)\eta(x) + r(x) = 0. \tag{3}$$

It has two real solutions $\eta_1(x) < 0$ and $\eta_2(x) > 0$, which characterize the layers at $x = 0$ and $x = 1$, respectively. Let:

$$\psi_1 = -\max_{x\in[0,1]}\eta_1(x) \quad \text{and} \quad \psi_2 = \min_{x\in[0,1]}\eta_2(x).$$

The situations of the two external layers are characterized by $\varepsilon_c^2 \ll \varepsilon_d$ or $\varepsilon_c^2/\varepsilon_d \to 0$ as $\varepsilon_c \to 0$, which imply that $\psi_1 \approx \psi_2 \approx \sqrt{\frac{\bar{p}}{\varepsilon_d}}$ we have the layers similar to the reaction–diffusion case $\varepsilon_c = 0$. Herein, a priori bounds for the solution and its derivatives are established, as follows:

Lemma 1. *For any $0 < d < 1$, we have up to a certain order q that it depends on the smoothness of the data*

$$|y^k(x)| \leqslant C\{1 + \psi_1^k e^{-d\psi_1 x} + \psi_2^k e^{-d\psi_2(1-x)}\}, \quad \text{for } 0 \leqslant k \leqslant q. \tag{4}$$

For the proof of the above lemma refer to Kadalbajoo and Yadaw (2008).

Lemma 2. *The solution $y(x)$ of Eqs. (1) and (2) has the representation, see (Linß and Roos, 2004)*

$$y(x) = u(x) + w_0(x) + w_1(x), \tag{5}$$

where

$$|u^{(k)}(x)| \leqslant C \qquad \text{for } 0 \leqslant k \leqslant q,$$
$$\left|w_0^{(k)}(x)\right| \leqslant C\psi_1^k e^{-b\psi_1 x} \quad \text{for } 0 \leqslant k \leqslant q,$$

and

$$\left|w_1^{(k)}(x)\right| \leqslant C\psi_2^k e^{-b\psi_2(1-x)} \quad \text{for } 0 \leqslant k \leqslant q.$$

3. Mesh selection strategy

In this section, for the selection of the mesh for the previously discussed three subintervals; the solution regions are provided. It is known that on an equidistant mesh no scheme can attain convergence at all mesh points uniformly in ε_c and ε_d, unless its coefficients have an exponential property. Therefore, unless a specially chosen mesh is used, we cannot obtain a parameter-uniform convergence at all the mesh points. The simple possible non-uniform mesh, namely a piecewise-uniform mesh discussed by Linß and Roos (2004), is sufficient for the construction of a parameter-uniform method. It is fine near layers but coarser otherwise. We do not claim that these piecewise-uniform meshes are optimal in any sense. It is attractive because of its simplicity and adequacy for handling a wide variety of singularly perturbed problems. The Shishkin mesh one should have a priori knowledge about the location and nature of the layers which, applicable only by using. To obtain the discrete counterpart of the two-parameter singularly perturbed boundary value problems Eqs. (1) and (2), firstly the considered mesh discretized the domain $\overline{\Omega} = [0,1]$ into three subintervals:

$$\Lambda_0 = [0,\gamma_1], \quad \Lambda_c = [\gamma_1, 1-\gamma_2] \quad \text{and} \quad \Lambda_1 = [1-\gamma_2, 1],$$

where transition parameters are given by

$$\gamma_1 = \min\left(\frac{1}{4}, \frac{2}{\psi_1}\ln n\right), \quad \gamma_2 = \min\left(\frac{1}{4}, \frac{2}{\psi_2}\ln n\right),$$

with n to be the number of subdivision points of the interval $[0,1]$ and we place $n/4$, $n/2$ and $n/4$ mesh points, respectively, in $[0,\gamma_1]$, $[\gamma_1, 1-\gamma_2]$ and $[1-\gamma_2, 1]$. Denote the step sizes in each subinterval by $h_1 = \frac{4\gamma_1}{n}, h_2 = \frac{2(1-\gamma_1-\gamma_2)}{n}$ and $h_3 = \frac{4\gamma_2}{n}$, respectively. Accordingly the resulting piecewise-uniform Shishkin mesh may be represented by:

$$\tilde{h} = \begin{cases} h_1 = \frac{4\gamma_1}{n}, & x_i = x_{i-1} + h_1, \text{ for } i = 1,2,\ldots,n/4, \\ h_2 = \frac{2(1-\gamma_1-\gamma_2)}{n}, & x_i = x_{i-1} + h_2, \text{ for } i = n/4+1,\ldots,3n/4, \\ h_3 = \frac{4\gamma_2}{n}, & x_i = x_{i-1} + h_3, \text{ for } i = 3n/4+1,\ldots,n. \end{cases}$$

4. Description of the current method

Consider a uniform mesh Δ with nodal point x_i on the interval $[0,1]$ such that Δ: $0 = x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ where

$$x_i = ih \quad \text{and} \quad h = \frac{1}{n}, \ i = 0,1,2,\ldots,n. \tag{6}$$

Let $y(x)$ be the exact solution of the problem presented by Eqs. (1) and (2) and S_i be an approximation solution to $y_i = y(x_i)$ obtained by the segment $Q_i(x)$ passing through the points

(x_i, S_i) and (x_{i+1}, S_{i+1}). Each mixed Spline segment $Q_i(x)$ has the following form, (for more details see (Zahra, 2009, 2011,)):

$$Q_i(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i(x - x_i) + d_i \quad i = 0, 1, 2, \ldots, n, \tag{7}$$

where a_i, b_i, c_i and d_i are constants and k is a free parameter.

To obtain the necessary conditions for the coefficients introduced in Eq. (7), the segment values of $Q_i(x_i), Q_i(x_{i+1}), Q_i^{(1)}(x_i)$and $Q_i^{(1)}(x_{i+1})$ should be considered at the common node. Expressions for the four coefficients of (7) can be developed in terms of S_i, S_{i+1}, M_i, and M_{i+1}, by defining:

$$Q_i(x_i) = S_i, \quad Q_i(x_{i+1}) = S_{i+1}, \quad Q_i^{(2)}(x_i) = M_i, \quad Q_i^{(2)}(x_{i+1}) = M_{i+1}. \tag{8}$$

Via a straightforward calculation we obtain the following expressions:

$$a_i = \frac{h^2(M_{i+1} - e^{-\theta}M_i)}{2\theta^2 \sinh(\theta)}, \quad b_i = \frac{h^2(e^{\theta}M_i - M_{i+1})}{2\theta^2 \sinh(\theta)},$$
$$c_i = \frac{(S_{i+1} - S_i)}{h} - \frac{h(M_{i+1} - M_i)}{\theta^2} \quad \text{and} \quad d_i = S_i - \left(\frac{h^2 M_i}{\theta^2}\right), \tag{9}$$

where $\theta = kh$ and $i = 0, 1, 2, \ldots, n$.

Using the continuity of the first derivative at the point (x_i, S_i), where $Q'_{i-1}(x)$ and $Q'_i(x)$ the following relation for $i = 1,2, \ldots, n-1$ is obtained;

$$(S_{i+1} - 2S_i + S_{i-1}) = h^2(\alpha M_{i+1} + \beta M_i + \alpha M_{i-1}), \tag{10}$$

where

$$\alpha = (\sinh(\theta) - \theta)/\theta^2 \sinh(\theta) \quad \text{and}$$
$$\beta = (2\theta\cosh(\theta) - 2\sinh(\theta))/\theta^2 \sinh(\theta),$$

when $k \to 0$ that $\theta \to 0$ then $(\alpha, \beta) = \frac{1}{6}(1, 4)$ and the relation defined by Eq. (10) reduces to the following ordinary cubic spline relation:

$$(S_{i+1} - 2S_i + S_{i-1}) = \frac{h^2}{6}(M_{i+1} + 4M_i + M_{i-1}), \tag{11}$$

at the point x_i the proposed singularly perturbed problem may be discretized by:

$$M_i = \frac{1}{\varepsilon_d}\left(\varepsilon_c p_i S_i^{(1)} + f_i\right), \tag{12}$$

where

$$S_i^{(1)} = \frac{S_{i+1} - S_{i-1}}{2h}, \quad S_{i+1}^{(1)} = \frac{3S_{i+1} - 4S_i + S_{i-1}}{2h},$$
$$S_{i-1}^{(1)} = \frac{-S_{i+1} + 4S_i - 3S_{i-1}}{2h}, \quad f_i = f(x_i, y_i) \quad \text{and} \quad p_i = p(x_i).$$

Substituting Eq. (12) into Eq. (11), we get the following nonlinear equations;

$$-\varepsilon_d(S_{i-1} - 2S_i + S_{i+1}) + \frac{\varepsilon_c h}{2}[D_i S_{i-1} + E_i S_i + A_i S_{i+1}]$$
$$= -h^2(\alpha f_{i-1} + \beta f_i + \alpha f_{i+1}) \quad i = 1, 2, ..., n-1. \tag{13}$$

Eq. (10) gives $n-1$ linear algebraic equations in $n-1$ unknowns S_i, where:

$$A_i = -\alpha p_{i-1} + \beta p_i + 3\alpha p_{i+1}, \quad D_i = -3\alpha p_{i-1} - \beta p_i + \alpha p_{i+1},$$
$$E_i = 4\alpha(-p_{i+1} + p_{i-1}).$$

5. Convergence analysis

In this section, the convergence analysis of the current method is investigated. The exponential spline solution of Eqs. (1) and (2) is based on the nonlinear equation given by Eq. (13). It can easily be seen that the system given by Eq. (13) gives $n-1$ nonlinear algebraic equations in the $n-1$ unknowns S_i,$i = 1, 2, \ldots, n-1$. This can be written in the standard matrix equation as:

$$N(S) = C, \tag{14}$$

where $N(S) = BS + h^2\widetilde{B}F(S)$ and the matrices B and $\tilde{B}$ may be written as:

$$B = \varepsilon_d B_0 + \frac{\varepsilon_c h}{2}B_1, \quad F = diag((f_y)_i), \tag{15}$$

where

$$B_0 = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{16}$$

$$\widetilde{B} = \begin{pmatrix} \beta & \alpha & & & \\ \alpha & \beta & \alpha & & \\ & & \ddots & & \\ & & \alpha & \beta & \alpha \\ & & & \alpha & \beta \end{pmatrix}, \tag{17}$$

$$B_1 = \begin{pmatrix} E_1 & A_1 & & & \\ D_2 & E_2 & A_2 & & \\ & & \ddots & & \\ & & D_{n-2} & E_{n-2} & A_{n-2} \\ & & & D_{n-1} & E_{n-1} \end{pmatrix}, \tag{18}$$

$$C_i = \begin{cases} \varepsilon_d\mu_1 - h^2\alpha f_0 - \frac{\varepsilon_c h}{2}D_1\mu_1, & i = 1, \\ 0, & i = 2, 3, \ldots, n-2, \\ \varepsilon_d\mu_2 - h^2\alpha f_n - \frac{\varepsilon_c h}{2}D_n\mu_2, & i = n-1. \end{cases} \tag{19}$$

Let $|p_i| \leqslant P$, $|F| \leqslant \widetilde{F}$ and let $N_{(i,j)}$ be the (i,j)th element of matrix N given by Eq. (14). Thus the row sums of the N satisfy:

$$N_i = \sum_j n_{1,j} = \varepsilon_d + \frac{\varepsilon_c h}{2}(2\alpha + \beta)P + h^2(2\alpha + \beta)\tilde{F}, \quad i = 1, n-1,$$
$$N_i = \sum_j n_{i,j} == h^2(2\alpha + \beta)\widetilde{F}, \quad i = 2, 3, \ldots, n-2,$$

for small values of ε_d and ε_c then matrix N is irreducible and monotone and it follows that N^{-1} exists and $N^{-1} \geqslant 0$ thus the system in Eq. (14) has a unique solution; see (Henrici, 1962).

Theorem 1. *Let $y(x)$ be the solution of Eqs. (1) and (2) and $S(x)$ be the solution the scheme defined on the piecewise-uniform Shishkin mesh. Then*

$$\|S(x)-y(x)\|_\infty \leqslant C(n^{-1}\ln n)^2, \tag{20}$$

where C is a constant independent of ε_d and ε_c.

Proof. The estimate is obtained on each subinterval $\Omega_i = [0,1]$ separately. Using (Kadalbajoo and Gupta, 2009), the ε-uniform error estimate is

$$|S(x)-y(x)| \leqslant Ch_i^2 \max_{\Omega_i}|y''(x)|. \tag{21}$$

In our analysis, we split the numerical solution S corresponding to the decomposition of the exact solution $S = U + W_0 + W_1$. Then from Lemma 1, on Ω_i, we get

$$\begin{aligned}|S(x)-y(x)| &\leqslant Ch_i^2 \max_{\Omega_i}|y''(x)|,\\ &\leqslant Ch_i^2\{1+\psi_1^2 e^{-j\psi_1 x}+\psi_2^2 e^{-j\psi_2(1-x)}\},\\ &\leqslant Ch_i^2\{1+\psi_1^2+\psi_2^2\}.\end{aligned} \tag{22}$$

Also, using Lemma 2 and Eq. (21) on Ω_i, see (Linß and Roos, 2004)

$$\begin{aligned}|S(x)-y(x)| &\leqslant |U+W_0+W_1-u-w_0-w_1|,\\ &\leqslant |U-u|+|W_0-w_0|+|W_1-w_1|,\\ &\leqslant Ch_i^2\max_{\Omega_i}|u''(x)|+2\max_{\Omega_i}|w_0(x)|+2\max_{\Omega_i}|w_1(x)|,\\ &\leqslant C(h_i^2+e^{-j\psi_1 x_i}+e^{-j\psi_2(1-x_i)}).\end{aligned} \tag{23}$$

Case 1 The argument now depends on whether $\frac{2\ln n}{\psi_1} \leqslant \frac{1}{4}$ and $\frac{2\ln n}{\psi_2} \leqslant \frac{1}{4}$. In this case$\psi_1 \leqslant C$ ln$n$ and $\psi_2 \leqslant C$ ln$n$. Then the result follows at once from Eq. (22)

$$|S(x)-y(x)| \leqslant C(n^{-1}\ln n)^2.$$

Case 2 When $\gamma_1 = \frac{2}{\psi_1}\ln n$ and $\gamma_2 = \frac{2}{\psi_2}\ln n$. Suppose that i satisfies $1 \leqslant i \leqslant \frac{n}{4}$ and $\frac{3n}{4} \leqslant i \leqslant n$. Then $h_i = \frac{4\gamma_1}{n} = \frac{Cn^{-1}\ln n}{\psi_1}$ and $h_i = \frac{4\gamma_2}{n} = \frac{Cn^{-1}\ln n}{\psi_2}$ respectively. Now from Eq. (22), we get

$$|S(x)-y(x)| \leqslant C(n^{-1}\ln n)^2,$$

if i satisfied $\frac{n}{4} \leqslant i \leqslant \frac{3n}{4}$. Then $\gamma_1 \leqslant x_i$ and $x_i \leqslant 1-\gamma_2$ or $\gamma_2 \leqslant 1-x_i$ and so

$$e^{-\psi_1 x_i} \leqslant e^{-\psi_1\gamma_1} = e^{-2\ln n} = n^{-2}, \quad \text{since } \gamma_1 = \frac{2\ln n}{\psi_1}.$$

$$e^{-\psi_2(1-x_i)} \leqslant e^{-\psi_1\gamma_2} = e^{-2\ln n} = n^{-2}, \quad \text{since } \gamma_2 = \frac{2\ln n}{\psi_2}.$$

Using this in the Eq. (23) gives the required result. □

6. Numerical examples

In this section, we apply our method to the following example and verify experimentally uniform convergence.

Example 1. Consider the singularly perturbed boundary value problem see (Kadalbajoo and Yadaw, 2008);

$$-\varepsilon_d y^{(2)}+\varepsilon_c y^{(1)}+y=\cos(\pi x), \quad x\in(0,1), \; y(0)=0, \; y(1)=0. \tag{24}$$

The exact solution is given by

$$y(x)=\rho_1\cos(\pi x)+\rho_2\sin(\pi x)+\psi_1\exp(\lambda_1 x)+\psi_2\exp(-\lambda_2(1-x)), \tag{25}$$

where

Table 1 Comparison of maximum errors for Example 1 with equidistant mesh.

ε_c	$\varepsilon_d = 10^{-2}$, $n = 128$			$\varepsilon_d = 10^{-4}$, $n = 128$		
	Kadalbajoo and Yadaw (2008)	Our methods		Kadalbajoo and Yadaw (2008)	Our methods	
		$\alpha=\frac{1}{12},\beta=\frac{10}{12}$	$\alpha=\frac{1}{24},\beta=\frac{22}{24}$		$\alpha=\frac{1}{12},\beta=\frac{10}{12}$	$\alpha=\frac{1}{24},\beta=\frac{22}{24}$
10^{-3}	8.3832-5	9.2993-7	4.1924-5	9.4446-3	1.3294-3	4.7598-3
10^{-4}	8.2686-5	1.1557-7	4.1296-5	9.0436-3	3.6708-4	4.2856-3
10^{-5}	8.2572-5	3.4933-8	4.1232-5	9.0036-3	2.8085-4	4.2295-3
10^{-6}	8.2561-5	2.6878-8	4.1226-5	8.9996-3	2.7232-4	4.2238-3
10^{-7}	8.2559-5	2.6072-8	4.1225-5	8.9992-3	2.7147-4	4.2232-3

Table 2 Comparison of maximum errors of Example 2, with equidistant mesh, $n = 128$, $\alpha = 1/12$, $\beta = 10/12$, and $\varepsilon_c = 1$.

x	$\varepsilon_d = 0.01$		$\varepsilon_d = 0.0015$	
	Lin et al. (2009)	Our method	Lin et al. (2009)	Our method
1/16	7.3-3	2.8-7	7.4-3	4.6-9
2/16	6.9-3	5.3-7	6.9-3	8.7-9
4/16	6.1-3	9.4-7	6.1-3	1.5-8
6/16	5.4-3	1.2-6	5.4-3	2.0-8
12/16	3.7-3	1.7-6	3.7-3	2.7-8
14/16	3.3-3	7.3-7	3.3-3	2.8-8

Table 3 Comparison of maximum errors for example 3 with equidistant mesh, $n = 1024$, $\alpha = 1/12$, $\beta = 10/12$, and $\varepsilon_c = 1$.

x	$\varepsilon_d = 0.01$		$\varepsilon_d = 0.0015$	
	Lin et al. (2009)	Our method	Lin et al. (2009)	Our method
100/1024	3.0-3	2.6976-3	1.2-3	3.5060-4
200/1024	2.5-3	2.1497-3	1.1-3	2.9906-4
300/1024	2.1-3	1.6944-3	1.0-3	2.4419-4
400/1024	1.8-3	1.3163-3	9.0-4	1.9313-4
500/1024	1.5-3	1.0014-3	8.0-4	1.4820-4
600/1024	1.3-3	7.3780-4	7.0-4	1.0962-4
700/1024	1.1-3	5.1589-4	7.0-4	7.6771-5
800/1024	9.0-4	3.2785-4	6.0-5	4.8823-5
900/1024	7.0-4	1.6753-4	5.0-4	2.4958-5
1000/1024	5.0-4	3.0047-5	5.0-4	4.4775-6

Table 4 Comparison of maximum errors for Example 4 with equidistant mesh, $\alpha = 1/12$, $\beta = 10/12$, and $\varepsilon_c = 1$.

n	$\varepsilon_d = 10^{-3}$		$\varepsilon_d = 10^{-4}$	
	Rao and Kumar (2008)	Our method	Rao and Kumar (2008)	Our method
64	3.999-4	4.526-5	3.859-3	3.815-3
128	1.002-4	2.848-6	9.937-5	2.716-4
256	2.509-5	1.783-7	2.503-5	1.751-5
512	6.274-6	1.115-8	6.270-6	1.113-6
1024	1.568-6	6.970-10	1.568-6	6.966-8

$$\rho_1 = \frac{\varepsilon_d\pi^2 + 1}{\varepsilon_c^2\pi^2 + (\varepsilon_d\pi^2 + 1)^2}, \quad \rho_2 = \frac{\varepsilon_c\pi}{\varepsilon_c^2\pi^2 + (\varepsilon_d\pi^2 + 1)^2},$$

$$\psi_1 = -\rho_1\frac{1 + \exp(-\lambda_2)}{1 - \exp(\lambda_1 - \lambda_2)}, \psi_2 = \rho_1\frac{1 + \exp(\lambda_1)}{1 - \exp(\lambda_1 - \lambda_2)},$$

$$\lambda_1 = \frac{\varepsilon_c - \sqrt{\varepsilon_c^2 + 4\varepsilon_d}}{2\varepsilon_d}, \quad \lambda_2 = \frac{\varepsilon_c + \sqrt{\varepsilon_c^2 + 4\varepsilon_d}}{2\varepsilon_d}.$$

Example 2. Consider the singularly perturbed boundary value problem see (Lin et al., 2009);

$$-\varepsilon_d y^{(2)} + y^{(1)} + y = 1, \quad x \in (0,1), \; y(0) = 0, \; y(1) = 0. \tag{26}$$

The exact solution is given by

$$y(x) = (e^{\lambda_2} - 1)e^{\lambda_1 x}/(e^{\lambda_1} - e^{\lambda_2}) + (1 - e^{\lambda_1})e^{\lambda_2 x}/(e^{\lambda_1} - e^{\lambda_2}) + 1, \tag{27}$$

where $\lambda_1 = \frac{1+\sqrt{1+4\varepsilon_d}}{2\varepsilon_d}, \lambda_2 = \frac{1-\sqrt{1+4\varepsilon_d}}{2\varepsilon_d}$.

Example 3. Consider the semi-linear singularly perturbed boundary value problem see (Lin et al., 2009);

$$\varepsilon_d y^{(2)} + 2y^{(1)} = -e^{y}, \quad x \in (0,1), \; y(0) = 0, \; y(1) = 0. \tag{28}$$

The exact solution is given by:

$$y(x) = (1 - e^{-2x/\varepsilon_d})\ln 2 - \ln(x+1). \tag{29}$$

Example 4. Consider the one-parameter singularly perturbed boundary value problem see (Rao and Kumar, 2008);

$$-\varepsilon_d y^{(2)} + y = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x), \quad x \in (0,1), \; y(0) = 0, \; y(1) = 0. \tag{30}$$

The exact solution is given by

$$y(x) = \frac{(e^{-(1-x)/\sqrt{\varepsilon_d}}) + e^{-x/\sqrt{\varepsilon_d}})}{1 + e^{-1/\sqrt{\varepsilon_d}}} - \cos^2(\pi x). \tag{31}$$

The estimated maximum pointwise error using exponential spline method applied to example1 is shown in Table 6. To compute the experimental rates of convergence Ord^n for every fixed ε_c and ε_d, we use the rate of convergence from:

Table 5 Comparison of maximum errors for example 1 with Shishkin mesh, $n = 128$, $\alpha = 1/12$ and $\beta = 10/12$.

ε_c	$\varepsilon_d = 10^{-6}$		$\varepsilon_d = 10^{-10}$	
	Kadalbajoo and Yadaw (2008)	Our method	Kadalbajoo and Yadaw (2008)	Our method
10^{0}	6.1243-3	6.5929-3	6.1108-3	6.8146-3
10^{-1}	1.9416-2	6.1326-3	1.9424-2	6.1407-3
10^{-2}	1.8314-2	5.7912-3	1.8500-2	6.4540-3
10^{-3}	1.3075-2	1.4464-3	1.8359-2	2.8582-3
10^{-4}	9.4539-3	1.7624-3	1.8163-2	2.8119-3
10^{-5}	9.0525-3	1.7689-3	1.3076-2	1.2664-3
10^{-6}	9.0124-3	1.7694-3	9.4540-3	7.8644-4
10^{-7}	9.0084-3	1.7695-3	9.0526-3	8.0674-4
10^{-8}	9.0080-3	1.7695-3	9.0125-3	8.0870-4
10^{-9}	9.0079-3	1.7695-3	9.0085-3	8.0889-4
10^{-10}	9.0079-3	1.7695-3	9.0081-3	8.0891-4
10^{-11}	9.0079-3	1.7695-3	9.0080-3	8.0892-4
10^{-12}	9.0079-3	1.7695-3	9.0080-3	8.0892-4

$$Ord^n = \frac{\ln E^n - \ln E^{2n}}{\ln 2}$$

The numerical results presented in Tables 1–4 clearly indicate that the proposed scheme with uniform mesh is not uniformly convergent for sufficiently small value of ε_c and ε_d and the maximal nodal error increases as the number of mesh points increases as in Table 4. To overcome this drawback, we have used a special piecewise uniform mesh known as Shishkin mesh. The numerical results displayed in Tables 5 and 6 clearly indicate that the proposed method based on exponential spline with Shishkin mesh is ε-uniformly convergent. Fig. 1 shows the exact and the approximate solution for various values of $\varepsilon_d = 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5} and for fix $\varepsilon_c = 10^{-6}$. Also, we note as $\varepsilon_d$decreases for fixed ε_c the width of boundary layer decreases and becomes more and more stiff at $x = 0$ and $x = 1$, this shows clearly the effect of ε_d and $\varepsilon_c$on the boundary layer (see Table 6). Fig. 2 shows the exact and approximate solutions which are taking the same shape and behavior. Also, numerical results generated by the proposed scheme indicate that the maximal nodal errors are smaller than those obtained by Lin et al. (2009), Rao and Kumar (2008), Rao et al. (2010), Kadalbajoo and Yadaw (2008) and Roos and Uzelac (2003).

7. Conclusion

A numerical method is developed to solve two-parameter singularly perturbed semi-Linear boundary value problems given by Eqs. (1) and (2). This method is based on exponential spline with a piecewise uniform Shishkin mesh. The method is shown to be uniformly convergent independent of mesh parameters and perturbation parameters ε_c and ε_d. It has been found that the proposed algorithm gives highly accurate numerical results and higher order of convergence than other existing methods.

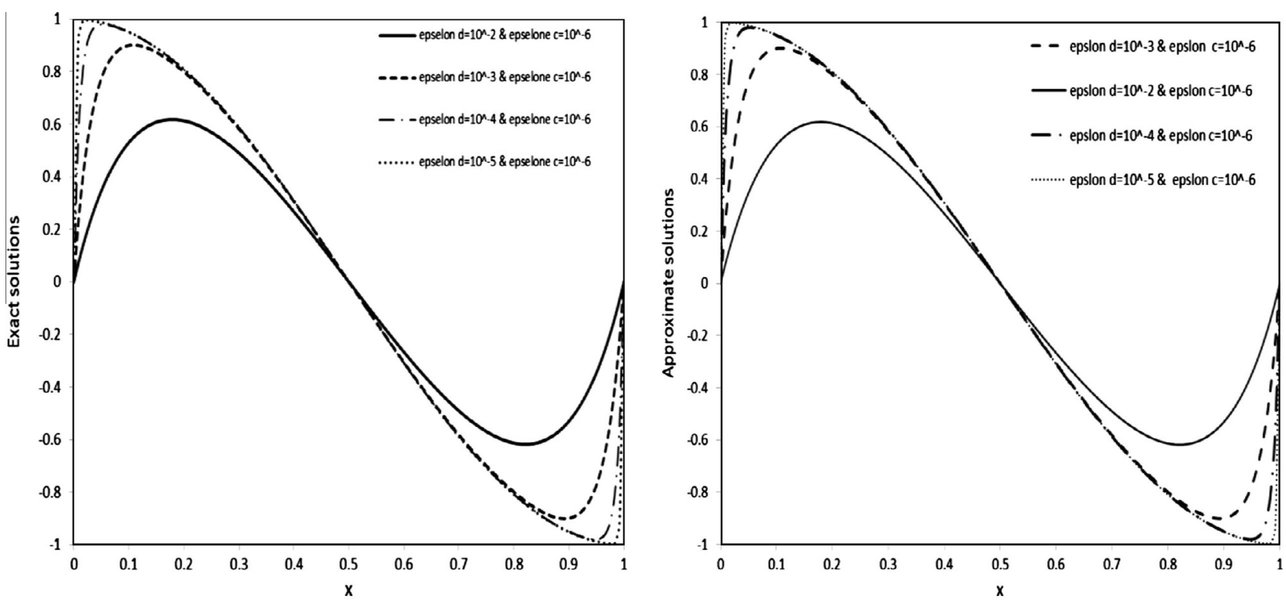

Figure 1 Exact and approximate solutions for example1 at different values of $\varepsilon_d = 10^{-2}$, 10^{-4}, 10^{-3}, 10^{-5} and for fix $\varepsilon_c = 10^{-6}$.

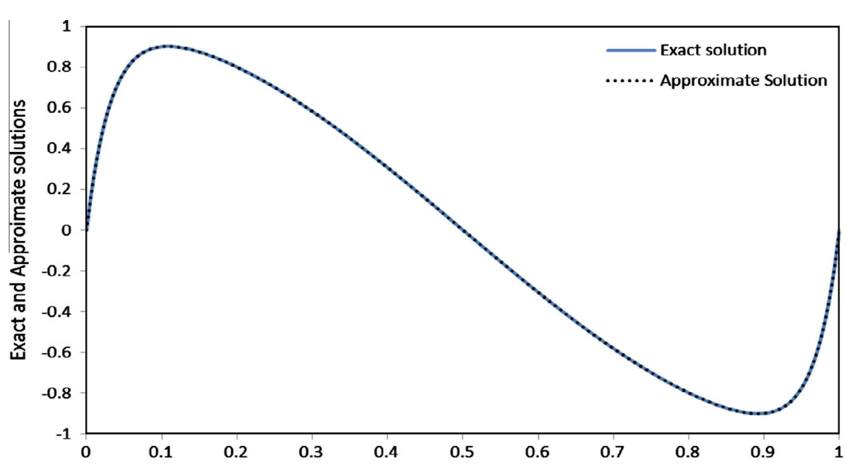

Figure 2 Exact and Approximate solutions for example1 at $\varepsilon_d = 10^{-3}$ and $\varepsilon_c = 10^{-6}$.

Table 6 Comparison of maximum errors and order of convergence for example 1 with Shishkin mesh, $n = 1024$, $\alpha = 1/12$ and $\beta = 10/12$.

ε_c	$\varepsilon_d = 10^{-2}$		$\varepsilon_d = 10^{-4}$		$\varepsilon_d = 10^{-6}$	
	Our method $n = 512$	Our method $n = 1024$	Our method $n = 512$	Our method $n = 1024$	Our method $n = 512$	Our method $n = 1024$
10^{-1}	1.1976-5	2.9941-6	5.3351-4	2.6639-4	1.3240-3	6.4629-4
	1.9999		1.0020		1.0346	
10^{-2}	6.1615-7	1.5401-7	4.9463-3	3.9897-3	1.1624-3	4.5526-4
	2.0003		0.3101		1.3523	
10^{-3}	5.6601-8	1.4131-8	6.7619-3	6.2360-3	6.4687-4	4.8213-4
	2.0020		0.1168		0.4240	
10^{-4}	5.7018-9	1.4064-9	6.6922-3	6.1992-3	4.8827-4	3.1132-4
	2.0191		0.1104		0.6493	
10^{-5}	6.6095-10	1.4622-10	6.6833-3	6.1933-3	4.7194-4	2.9663-4
	2.1764		0.1098		0.6699	
10^{-6}	1.5743-10	2.0296-11	6.6824-3	6.1927-3	4.7033-4	2.9520-4
	2.9554		0.1098		0.6720	
10^{-7}	1.0712-10	7.7557-12	6.6823-3	6.1926-3	4.7017-4	2.9506-4
	3.7878		0.1098			
10^{-8}	1.0209-10	6.4920-12	6.6823-3	6.1926-3	4.7016-4	2.9505-4
	3.9750		0.1098		0.6722	
10^{-9}	1.0160-10	6.3608-12	6.6823-3	6.1926-3	4.7015-4	2.9504-4
	3.9975		0.1098		0.6722	
10^{-10}	1.0154-10	6.3671-12	6.6823-3	6.1926-3	4.7015-4	2.9504-4
	3.9953		0.1098		0.6722	
10^{-11}	1.0154-10	6.3508-12	6.6823-3	6.1926-3	4.7015-4	2.9504-4
	3.9953		0.1098		0.6722	
10^{-12}	1.0153-10	6.3532-12	6.6823-3	6.1926-3	4.7015-4	2.9504-4
	3.9953		0.1098		0.6722	
	$\varepsilon_d = 10^{-8}$		$\varepsilon_d = 10^{-10}$		$\varepsilon_d = 10^{-12}$	
10^{-1}	1.3325-3	6.5505-4	1.3326-3	6.5549-4	1.3267-3	6.5136-4
	1.0244		1.0236		1.0263	
10^{-2}	1.5181-3	7.4409-4	1.5219-3	7.4713-4	1.5219-3	7.4716-4
	1.0287		1.0264		1.0264	
10^{-3}	2.9398-4	1.0464-4	4.3265-4	2.0375-4	4.3430-4	2.0500-4
	1.4903		1.0864		1.083	
10^{-4}	1.3035-4	4.8235-5	2.8760-4	8.8530-5	2.9368-4	9.0505-5
	1.4342		1.6998		1.6982	
10^{-5}	9.9913-5	3.9352-5	1.3035-4	4.0206-5	2.8793-4	8.8692-5
	1.3442		1.6969		1.6988	
10^{-6}	9.9090-5	3.8527-5	6.2985-5	1.7731-5	1.3035-4	4.0192-5
	1.3629		1.8287		1.6974	
10^{-7}	9.9006-5	3.8446-5	6.3354-5	1.7717-5	5.9287-5	1.5575-5
	1.3647		1.8383		1.9285	
10^{-8}	9.8998-5	3.8438-5	6.3388-5	1.7716-5	5.9771-5	1.5638-5
	1.3649		1.8392		1.9344	
10^{-9}	9.8997-5	3.8437-5	6.3392-5	1.7716-5	5.9816-5	1.5644-5
	1.3649		1.8392		1.9349	
10^{-10}	9.8997-5	3.8437-5	6.3392-5	1.7716-5	5.9819-5	1.5645-5
	1.3649		1.8392		1.9349	
10^{-11}	9.8997-5	3.8437-5	6.3392-5	1.7716-5	5.9820-5	1.5645-5
	1.3649		1.8392		1.9349	
10^{-12}	9.8997-5	3.8437-5	6.3392-5	1.7716-5	5.9821-5	1.5645-5
	1.3649		1.8392		1.9349	

References

Gracia, J.L., O'Riordan, E., Pickett, M.L., 2006. A parameter robust second order numerical method for a singularly, perturbed two-parameter problem. Appl. Numer. Math. 56, 962–980.

Henrici, P., 1962. Discrete Variable Methods in Ordinary Differential Equations. John Wiley, New York.

Kadalbajoo, M.K., Gupta, V., 2009. Numerical solution of singularly perturbed convection–diffusion problem using parameter uniform B-spline collocation method. J. Math. Anal. Appl. 355, 439–452.

Kadalbajoo, M.K., Yadaw, A.S., 2008. B-spline collocation method for a two-parameter singularly perturbed convection–diffusion boundary value problem. Appl. Math. Comput. 201, 504–513.

Lin, B., Li, K., Cheng, Z., 2009. B-spline solution of a singularly perturbed boundary value problem arising in biology. Chaos, Solitons and Fractals 42, 2934–2948.

Linß, T., Roos, H.G., 2004. Analysis of a finite-difference scheme for a singularly perturbed problem with two small parameters. J. Math. Anal. Appl. 289, 355–366.

Rao, R.N., Chakravarthy, P.P., 2012. A modified Numerov method for solving singularly perturbed differential–difference equations arising in science and engineering. Results Phys. 2, 100–103.

Rao, S.C.S., Kumar, M., 2008. Exponential B-spline collocation method for self -adjoint singularly perturbed boundary value problems. Appl. Numer. Math. 58, 1572–1581.

Rao, S.C.S., Kumar, S., Kumar, M., 2010. A parameter-uniform B-spline collocation method for singularly perturbed semilinear reaction–diffusion problems. J. Optim. Theory Appl. 146, 795–809.

Reddy, Y.N., Pramod, C.P., 2003. Method of reduction of order for solving singularly perturbed tow-point boundary value problems. Appl. Math. Comput. 136, 27–45.

Roos, H.G., Uzelac, Z., 2003. The SDFEM for a convection–diffusion problem with two small parameters. Comput. Methods Appl. Math. 3, 443–458.

Stynes, M., Kopteva, N., 2011. stabilised approximation of interior-layer solutions of a singularly perturbed semilinear reaction–diffusion problem. Numerische Mathematik 119, 787–810.

Valanarasu, T., Ramanujam, N., 2003. Asymptotic initial value methods for two-parameter singularly perturbed boundary value problems for second order ordinary differential equations. Appl. Math. Comput. 137, 549–570.

Valarmathi, S., Ramanujam, N., 2003. Computational methods for solving two-parameter singularly perturbed boundary value problems for second-order ordinary differential equations. Appl. Math. Comput. 136, 415–441.

Zahra, W.K., 2009. Exponential spline solutions for a class of two point boundary value problems over a semi-infinite range. Numer. Algor. 52, 561–573.

Zahra, W.K., 2011. Finite-difference technique based on exponential splines for the solution of obstacle problems. Int. J. Computer Math. 88 (14), 3046–3060.

Zahra, W.K., 2011. A smooth approximation based on exponential spline solutions for nonlinear fourth order two point boundary value problems. Appl. Math. Comput. 217, 8447–8457.